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A laminar swirled free-convection jet

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Abstract—The results of a theoretical analysis of the effect of free convection on the flow and heat transfer in vertical swirled plumes are presented. The cases of free convection acting in favour and against the forced convection are considered.

1. INTRODUCTION

AN EFFECTIVE way of solving a variety of problems of the applied aerodynamics is the use of swirled jet flows. The interest in swirled viscous jets is engendered by the fact that these, in contrast to straight ones, exhibit a centrifugal action which causes the transverse and longitudinal pressure gradients. The study of these phenomena is often of practical importance: for cooling turbo-electric generators, discs of gas turbines, for the operation of flow burners where flow swirling is used for flame stabilization—this is a far from being complete list of technical applications of the problems considered—whence a great variety of works concerned with the investigation of swirled viscous jets, whose rather detailed survey can be found in refs. [1–3].

The theory of viscous swirled jets goes back to work [4] where the first analytical solution of the problem was given. The presence of two quantities (K_0, L_0), characteristic of a swirled jet, prevents the basic equations from being reduced to one differential equation and makes the problem a non-self-similar one whose solution is sought with the aid of asymptotic-series expansions of the stream function in negative powers of the distance from the jet origin. The calculations begun by Loitsyansky [4] have been brought up to the fourth approximation for ψ, u, v and to the third for Δp, w [5]. At present the works [6, 7] continue to appear in the permanent literature that refine the mathematical aspect of the problem. The experimental material, which has been accumulated up to date [2, 5], but which seems to be far from exhausting the problem, generally indicates the high complexity inherent in the flow, as a result of which it lends itself with difficulty to theoretical analysis.

While the swirled viscous jets have been studied in great detail, the swirled free-convection jets, representing an interesting combined motion of a swirled and a free-convection jet, have hardly been studied at all. The reason for this is, first of all, that each of the flows considered (a swirled jet and a free-convection one) is a challenge in its own right.

The well known, self-similar solutions [8] for buoyant jets have been obtained presuming the equilibrium values of the Archimedes and inertial forces. However, the validity range of the formulae thus derived is limited only by large values of x where a jet degenerates into a free-convective flow. Moreover, the relations are valid only for the case when the direction of free convection-induced flow coincides with that of the liquid flow in the jet. As regards the literature on free convection in swirled vertical jets, very little seems to have been published [9, 10].

In view of this the object of this work was to theoretically investigate the effect of buoyancy forces on the flow and heat transfer in laminar vertical swirled jets.

2. GOVERNING EQUATIONS

2.1.

Consider a swirled viscous incompressible liquid jet issuing from a source and propagating vertically upward (downward) in an isothermal medium under the action of buoyancy forces and initial momentum of the jet ($T_0 > T_\infty$). Within the framework of the laminar boundary-layer model, the basic equations are written down in the cylindrical coordinate system (Fig. 1) as [11]:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial \Delta p}{\partial x}+\frac{\nu}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)\pm g\beta_q(\Delta T)^q \quad (1)$$

$$\frac{\partial}{\partial x}(yu)+\frac{\partial}{\partial y}(yv) = 0 \quad (2)$$

$$\frac{w^2}{y} = \frac{1}{\rho}\frac{\partial \Delta p}{\partial y} \quad (3)$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{vw}{y} = \nu\left(\frac{\partial^2 w}{\partial y^2}+\frac{1}{y}\frac{\partial w}{\partial y}-\frac{w}{y^2}\right) \quad (4)$$

$$u\frac{\partial \Delta T}{\partial x}+v\frac{\partial \Delta T}{\partial y} = \frac{k}{y}\frac{\partial}{\partial y}\left(y\frac{\partial \Delta T}{\partial y}\right). \quad (5)$$

A change in the medium density [12]

$$\rho = \rho_\infty[1-\beta_q(T-T_\infty)^q] \quad (6)$$

is taken into account only by the term corresponding to the buoyancy force, which represents the Boussinesq approximation [13–15]. The plus and minus signs in front of the last term on the RHS of equation (1) relate to

NOMENCLATURE

C_p	specific heat at constant pressure
d_0	nozzle diameter
E	kinetic energy flux, $E_1 = E/\mu u_0^2 d_0$
g	gravity acceleration
Gr	Grashof number, $g\beta_q d_0^3 \lvert T_0 - T_\infty \rvert q/v^2$
K_0	momentum flux, $K_1 = K_0/\mu u_0 d_0$
k	thermal diffusivity
L_0	momentum flux, $L_1 = L_0/(\mu^2 u_0 v d_0^3)^{1/2}$
$\dot{m}$	liquid flow rate in jet, $\dot{M} = m/\mu d_0$
Δp	excess pressure, $P = (p - p_\infty)/\rho u_0^2$
Pr	Prandtl number
Q_0	excess enthalpy flux, $Q_1 = Q_0/(T_0 - T_\infty) C_p \mu d_0$
Re	Reynolds number, $u_0 d_0/v$
ΔT	excess temperature, $\theta = \Delta T/(T_0 - T_\infty)$
u	axial velocity, $U = u/u_0$
v	radial velocity, $V = v(d_0/u_0 v)^{1/2}$
w	tangential velocity, $W = w/u_0$
x	axial coordinate, $X = x/d_0$
y	transverse coordinate, $Y = y(u_0/d_0 v)^{1/2}$.

Greek symbols

β_q, q	density parameters [12, 15]
β_1	coefficient of liquid volumetric expansion
ε	small parameter, $\pm 2 \dfrac{Gr}{Re^2} X^{3-q}$
μ	dynamic viscosity
ρ	density
v	kinematic viscosity
ψ	stream function.

Subscripts

0	jet at exit
∞	surrounding medium
m	maximum
T	at jet stagnation cross-section.

an ascending and descending forced flow, respectively.

The boundary conditions are

$$v = \partial u/\partial y = \partial \Delta T/\partial y = w = 0 \quad \text{at} \quad y = 0$$

$$u \to 0, \quad \Delta T \to 0, \quad w \to 0, \quad \Delta p \to 0 \quad \text{for} \quad y \to \infty. \tag{7}$$

The integral relations completing the problem formulation are

$$\frac{d}{dx} \int_0^\infty \left(u^2 + \frac{\Delta p}{\rho}\right) y \, dy = \pm g\beta_q \int_0^\infty (\Delta T)^q y \, dy$$

$$2\pi C_p \int_0^\infty \rho u \Delta T y \, dy = Q_0$$

$$2\pi \int_0^\infty \rho u w y^2 \, dy = L_0. \tag{8}$$

2.2.

For convenience, basic equations (1)–(5), boundary (7) and integral (8) conditions will be given in dimensionless form:

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{1}{Y} \frac{\partial}{\partial Y}\left(Y \frac{\partial U}{\partial Y}\right) \pm \frac{Gr}{Re^2} \theta_q \tag{9}$$

$$\frac{\partial}{\partial X}(YU) + \frac{\partial}{\partial Y}(YV) = 0 \tag{10}$$

$$\frac{W^2}{Y} = \frac{\partial P}{\partial Y} \tag{11}$$

$$U \frac{\partial W}{\partial X} + V \frac{\partial W}{\partial Y} + \frac{VW}{Y} = \frac{\partial^2 W}{\partial Y^2} + \frac{1}{Y} \frac{\partial W}{\partial Y} - \frac{W}{Y^2} \tag{12}$$

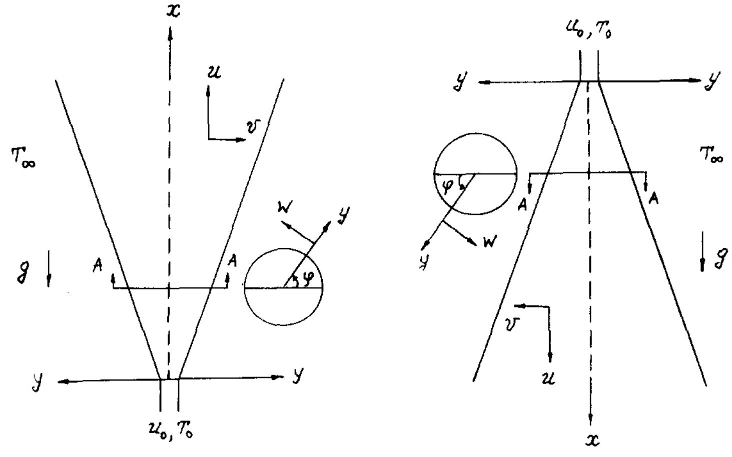

FIG. 1. Scheme of flow of a vertical swirled jet.

$$U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y}=\frac{1}{Pr}\left(\frac{\partial^2\theta}{\partial Y^2}+\frac{1}{Y}\frac{\partial\theta}{\partial Y}\right) \quad (13)$$

$$V=\partial U/\partial Y=\partial\theta/\partial Y=W=0 \quad \text{at} \quad Y=0 \quad (14)$$

$$U\to 0,\quad \theta\to 0,\quad W\to 0,\quad P\to 0 \quad \text{for} \quad Y\to\infty$$

$$2\pi\int_0^\infty U\theta Y\,dY=Q_1,\quad 2\pi\int_0^\infty UWY^2\,dY=L_1,$$

$$\frac{d}{dX}\int_0^\infty (U^2+P)Y\,dY=\pm\frac{Gr}{Re^2}\int_0^\infty \theta^q Y\,dY \quad (15)$$

3. SELF-SIMILAR SOLUTIONS

The most simple approach to the problems in the class of jet flows is the finding of the self-similar solutions for the boundary-layer equations. However, the inclusion of the buoyancy forces substantially complicates the problem, so that the self-similar solutions are possible only at fixed Prandtl numbers.

3.1. *Exact solutions*

Introduce the following dimensionless quantities: $\psi(X,\eta)$, $\theta(X,\eta)$, $W(X,\eta)$, and $P(X,\eta)$:

$$\psi=f_0(\eta)X,\quad \theta=h_0(\eta)X^{-1} \quad (16)$$
$$W=b_0(\eta)X^{-(5+q)/4},\quad P=d_0(\eta)X^{-(5+q)/2}$$

where the dimensionless coordinates are defined as

$$X=X,\quad \eta=\frac{Y^2}{4}X^{-(1+q)/2}. \quad (17)$$

The substitution of expressions (16)–(17) into equations (9), (11–(13) yields the following system of equations:

$$(\eta f_0'')'+\frac{1}{2}f_0f_0''-\frac{1}{4}(1-q)f_0'^2\pm 2\frac{Gr}{Re^2}h_0^q=0$$

$$\frac{1}{Pr}(\eta h_0')'+\frac{1}{2}(f_0h_0)'=0$$

$$(\eta b_0')'+\frac{1}{2}(f_0b_0)'-\frac{1}{4\eta}b_0(1-f_0)=0$$

$$2a_0'\eta=b_0^2 \quad (18)$$

subject to the boundary conditions

$$f_0(0)=0,\quad \lim_{\eta\to 0}\sqrt{\eta}f_0''=0,\quad f_0'(\infty)=0$$

$$\lim_{\eta\to 0}\sqrt{\eta}h_0'=0,\quad h_0(\infty)=0$$

$$b_0(0)=0,\quad b_0(\infty)=0,\quad a_0(\infty)=0. \quad (19)$$

Here the prime means differentiation with respect to η. The solution of equations (18), satisfying boundary conditions (19), is:

At $qPr=2$

$$f_0=\frac{4\alpha\eta}{1+\alpha\eta},\quad h_0=\frac{c_1}{(1+\alpha\eta)^{2Pr}},\quad b_0=\lambda\frac{\sqrt{\alpha}\eta^{1/2}}{(1+\alpha\eta)^2}$$

$$a_0=-\frac{\lambda^2}{6}\frac{1}{(1+\alpha\eta)^3},\quad c_1=\frac{(2Pr+1)Q_1}{8\pi},$$

$$\lambda=\frac{3L_1\sqrt{\alpha}}{8\pi},\quad \pm\frac{Gr}{Re^2}c_1^q=2\alpha^2(3-q). \quad (20)$$

At $4qPr=3+q$

$$f_0=\frac{24}{3+q}\frac{\alpha\eta}{1+\alpha\eta},\quad h_0=c_1(1+\alpha\eta)^{-12Pr/(3+q)}$$

$$b_0=\lambda\sqrt{\alpha}\eta^{1/2}(1+\alpha\eta)^{-12/(3+q)},$$

$$a_0=\frac{3+q}{2(q-21)}\lambda^2(1+\alpha\eta)^{q-21/(3+q)}$$

$$c_1=\frac{(12Pr+3+q)Q_1}{48\pi},\quad \lambda=\frac{L_1\sqrt{\alpha}(15+q)}{8(3+q)\pi} \quad (21)$$

$$\pm\frac{Gr}{Re^2}c_1^q=\frac{48(3-q)}{(3+q)^2}\alpha^2.$$

Note that the self-similar solutions (20), (21) obtained are independent of the initial jet momentum magnitude

$$2\pi\int_0^\infty U^2Y\,dY=K_1 \quad (22)$$

and, consequently, are valid far downstream from the source where the swirled jet degenerates into a free-convective flow. Moreover, the above formulae no longer hold for free and forced convection flows propagating in opposite directions.

3.2. *Approximate solution*

Consider an analytical solution of the system of differential equations (18) at $q=1$. The solution of the first two equations of system (18) subject to boundary conditions (19) will be sought in the form

$$f_0=c\frac{\alpha\eta}{1+\alpha\eta},\quad h_0=\frac{c_1}{(1+\alpha\eta)^{cPr/2}}. \quad (23)$$

The substitution of equations (23) into the first equation of system (18) will yield

$$6c\alpha^3\eta-2c\alpha^2(1+\alpha\eta)-c^2\alpha^3\eta$$
$$+2\frac{Gr}{Re^2}c_1(1+\alpha\eta)^{4-(cPr/2)}=0. \quad (24)$$

Equation (24) admits the solution of the form of equations (23) at $cPr/2=4$ and $cPr/2=3$. With this choice of the constants, $c=4$ and $Pr=2$, $c=6$ and $Pr=1$ [8]. To obtain the approximate solution, expand the last term of equation (24) into the series

$$(1+\alpha\eta)^{4-cPr/2}=1+\left(4-\frac{cPr}{2}\right)\alpha\eta+\cdots$$

and stop with two expansions. This gives

$$f_0 = \frac{12}{1+Pr}\frac{\alpha\eta}{1+\alpha\eta}, \quad h_0 = c_1(1+\alpha\eta)^{-6Pr/(1+Pr)}$$

$$c_1 = \frac{(7Pr+1)Q_1}{24\pi}, \quad c_1\frac{Gr}{Re^2} = \frac{12}{1+Pr}\alpha^2. \tag{25}$$

The integration of the third and fourth equations of system (18) yields

$$b_0 = \lambda\sqrt{\eta}\,\alpha^{1/2}(1+\alpha\eta)^{-6/1+Pr}, \quad \lambda = \frac{L_1\sqrt{\alpha}(Pr+7)}{8\pi(1+Pr)}$$

$$a_0 = \frac{1+Pr}{2(Pr-11)}\lambda^2(1+\alpha\eta)^{(Pr-11)/(1+Pr)}. \tag{26}$$

Consequently, the velocity and temperature fields in vertical axisymmetric swirled jets are described by the following approximate formulae

$$U = \frac{6}{1+Pr}\frac{\alpha}{(1+\alpha\eta)^2},$$

$$P = \frac{1+Pr}{2(Pr-11)}\lambda^2(1+\alpha\eta)^{(Pr-11)/(1+Pr)}X^{-3}$$

$$\theta = c_1(1+\alpha\eta)^{-6Pr/(1+Pr)}X^{-1},$$

$$W = \lambda\sqrt{\alpha}\eta^{1/2}(1+\alpha\eta)^{-6/(1+Pr)}X^{-3/2}. \tag{27}$$

Expressions (25) and (26) for $Pr = 2$ and $Pr = 1$ pass over, accurate to the constants α, c_1, λ, into formulae (20) and (21) at $q = 1$.

4. PERTURBATION METHOD

The system of coupled basic equations (9)–(13) contains four dependent quantities U, θ, P, W. The solution of this system at small values of the parameter ε can be presented in the form

$$\psi(X,\eta) = X\sum_{i=0}^{\infty} f_i(\eta)\varepsilon^i, \quad \theta(X,\eta) = X^{-1}\sum_{i=1}^{\infty} h_i(\eta)\varepsilon^i,$$

$$W(X,\eta) = X^{-2}\sum_{i=1}^{\infty} b_i(\eta)\varepsilon^i, \quad P(X,\eta) = X^{-4}\sum_{i=0}^{\infty} a_i(\eta)\varepsilon^i. \tag{28}$$

Here

$$X = X, \quad \eta = Y^2/4X^2 \tag{29}$$

Substituting expansions (28) into equations (9)–(13), written in terms of variables (29), and equating the terms with identical powers of ε, will give

$$(\eta f_0'')' + \frac{1}{2}(f_0 f_0')' = 0, \quad \frac{1}{Pr}(\eta h_0')' + \frac{1}{2}(f_0 h_0)' = 0, \quad (\eta b_0')' + \frac{1}{2}(f_0 b_0)' - \frac{1}{4\eta}b_0(1-f_0) = 0, \quad 2a_0'\eta = b_0^2 \tag{30}$$

$$(\eta f_i'')' + \frac{1}{2}f_0 f_i'' - \frac{1}{2}[(3-q)i-2]f_0' f_i' + \frac{1}{2}[(3-q)i+1]f_0'' f_i = A_i \tag{31}$$

$$\frac{1}{Pr}(\eta h_i')' + \frac{1}{2}f_0 h_i' - \frac{1}{2}[(3-q)i-1]f_0' h_i = B_i \tag{32}$$

$$(\eta b_i')' + \frac{1}{2}f_0 b_i' - \frac{1}{2}[(3-q)i-1]f_0' b_i - \frac{1}{4\eta}b_i(1-f_0) = C_i \tag{33}$$

$$2a_i'\eta = D_i \tag{34}$$

$$A_i = \sum_{j=1}^{i-1}\left\{\frac{1}{2}[(3-q)(i-j)-1]f_j' f_{i-j}' - \frac{1}{2}[(3-q)j+1]f_j f_{i-j}''\right\} - \Phi_i$$

$$B_i = \sum_{j=1}^{i-1}\left\{\frac{1}{2}[(3-q)(i-j)-1]f_j' h_{i-j} - \frac{1}{2}[(3-q)j+1]f_j h_{i-j}'\right\} - \frac{1}{2}f_i' h_0 - \frac{1}{2}[(3-q)i+1]f_i h_0'$$

$$C_1 = \sum_{j=1}^{i-1}\left\{\frac{1}{2}[(3-q)(i-j)-1]f_j' b_{i-j} - \frac{1}{2}[(3-q)j+1]f_j b_{i-j}' - \frac{(3-q)j+1}{4\eta}f_j b_{i-j}\right\}$$
$$-\frac{1}{2}f_i' b_0 - \frac{1}{2}[(3-q)i+1]f_i b_0' - \frac{(3-q)i+1}{4\eta}f_i b_0$$

$$D_i = 2b_i b_0 + \sum_{j=1}^{i-1} b_j b_{i-j}, \quad \Phi_1 = h_0^q, \quad \Phi_2 = qh_1 h_0^{q-1}, \quad \Phi_3 = qh_2 h_0^{q-1} + \frac{q(q-1)}{2}h_1^2 h_0^{q-2}, \ldots, \quad i = 1,2,3\ldots.$$

The boundary conditions are

$$f_i(0) = 0, \quad \lim_{\eta\to 0}\sqrt{\eta}f_i'' = 0, \quad f_i'(\infty) = 0, \quad \lim_{\eta\to 0}\sqrt{\eta}h_i' = 0, \quad h_i(\infty) = 0, \quad b_i(0) = 0, \quad b_i(\infty) = 0, \quad a_i(\infty) = 0 \tag{35}$$

The solution of equations (30) is known [1, 4]

$$f_0 = \frac{4\alpha\eta}{1+\alpha\eta}, \quad h_0 = \frac{c_1}{(1+\alpha\eta)^{2Pr}}, \quad b_0 = \frac{\sqrt{\alpha}\eta^{1/2}}{(1+\alpha\eta)^2}, \quad \alpha = \frac{3K_1}{16\pi},$$
$$a_0 = -\frac{\lambda^2}{6}\frac{1}{(1+\alpha\eta)^3}, \quad \lambda = \frac{3\sqrt{\alpha}L_1}{8\pi}, \quad c_1 = \frac{(2Pr+1)Q_1}{8\pi}. \tag{36}$$

In order to find the higher-order terms, use should be made of equation (31) in which it is advisable to pass over to the new variable $z = (1+\alpha\eta)^{-1}$ and to introduce the new function $y_i(z) = z^4 f_i'''(z)$. Then the following equation can be obtained for $y_i(z)$:

$$z(z-1)y_i'' + (1+2z)y_i' + 2(3-q)iy_i = z^3\left(\frac{A_i}{\alpha^2 z^3}\right)''.$$

Here the prime means differentiation with respect to z. The solution of this equation can be represented by the sum of the solutions of a corresponding differential equation and particular integral. It can be easily seen that a homogeneous equation is the equation of hypergeometric type which has no bounded solutions within the range $0 < z < 1$, since the coefficient of y_i is not among the eigenvalues of this equation [16]

$$-2(3-q)i = n(n+1), \quad i = 1,2,3\ldots, \quad n = 0,1,2\ldots.$$

Consequently, the only bounded solution of the homogeneous equation is $y_i \equiv 0$. Then the integration of equation (31) is reduced to finding its partial solution. Note that similar results are also obtained from the analysis of equations (32)–(34).

Thus, as the investigation shows, the search for the terms of expansion (28) of higher order ($i \geqslant 1$) is reduced to the finding of particular integrals of equations (31)–(34). It is hardly possible to write out their analytical expressions for a general case (for any Pr and q). At $qPr = 2$, the sought-after functions are expressed as (recurrent formulae):

$$f_{i+1} = \frac{1}{i+1}\frac{c_1^q}{8\alpha^2(3-q)}(f_i'\eta - 2if_i)$$
$$h_{i+1} = \frac{1}{i+1}\frac{c_1^q}{8\alpha^2(3-q)}(h_i'\eta - 2ih_i)$$
$$b_{i+1} = \frac{1}{i+1}\frac{c_1^q}{8\alpha^2(3-q)}\left(b_i'\eta + \frac{1}{2}(1-4i)b_i\right)$$
$$a_{i+1} = \frac{1}{i+1}\frac{c_1^q}{8\alpha^2(3-q)}(a_i'\eta + (1-2i)a_i)$$
$$i = 0,1,2\ldots. \tag{37}$$

These solutions satisfy the integral conditions, which in terms of variables (29) will be written as

$$(3-q)i\int_0^\infty\left(2f_0'f_i' + \sum_{j=1}^{i-1} f_j'f_{i-j}'\right)d\eta = 2\int_0^\infty \Phi_i\, d\eta$$
$$\int_0^\infty\left(f_0'h_i + \sum_{j=1}^{i} f_j'h_{i-j}\right)d\eta = 0$$
$$\int_0^\infty\left(f_0'b_i + \sum_{j=1}^{i} f_j'b_{i-j}\right)\eta^{1/2}\, d\eta = 0$$
$$i = 1,2,3\ldots. \tag{38}$$

On having collected the results obtained, it is possible to derive the following computational formulae for vertical swirled jets:

$$U = \frac{1}{2}\left\{f_0'X^{-1} + \frac{c_1^q}{8\alpha^2(3-q)}\sum_{i=0}^{\infty}\frac{1}{i+1} \times (f_i''\eta + (1-2i)f_i')\varepsilon^{1+i}X^{-1}\right\}$$

$$\theta = h_0X^{-1} + \frac{c_1^q}{8\alpha^2(3-q)}\sum_{i=0}^{\infty}\frac{1}{i+1} \times (h_i'\eta - 2ih_i)\varepsilon^{1+i}X^{-1}$$

$$W = b_0X^{-2} + \frac{c_1^q}{8\alpha^2(3-q)}\sum_{i=0}^{\infty}\frac{1}{i+1} \times \left(b_i'\eta + \frac{1}{2}(1-4i)b_i\right)\varepsilon^{1+i}X^{-2}$$

$$P = a_0X^{-4} + \frac{c_1^q}{8\alpha^2(3-q)}\sum_{i=0}^{\infty}\frac{1}{i+1} \times (a_i'\eta + (1-2i)a_i)\varepsilon^{1+i}X^{-4} \tag{39}$$

5. GENERALIZED SELF-SIMILAR SOLUTION

Consider an alternative method of deriving the functions of distribution of velocities, temperatures, swirling and pressure along the jet. The solution of the problem will be sought in the form

$$\psi(X,\eta) = f(\eta)s(X), \quad \theta(X,\eta) = h(\eta)\sigma(X)$$
$$W(X,\eta) = b(\eta)\omega(X), \quad P(X,\eta) = a(\eta)\kappa(X)$$
$$X = X, \quad \eta = \frac{Y^2}{4}\delta(X). \tag{40}$$

The application of transformation (40) to equations (9)–(13) will give

$$(\eta f_{\eta\eta})_\eta s\delta^2 + \frac{1}{2}ff_{\eta\eta}s_X s\delta^2 - \frac{1}{2}f_\eta^2 s\delta(s\delta)_X \pm 2\frac{Gr}{Re^2}(h\sigma)^q = 0$$
$$\frac{1}{Pr}(\eta h_\eta)_\eta\sigma\delta + \frac{1}{2}fh_\eta s_X\sigma\delta - \frac{1}{2}f_\eta hs\sigma_X\delta = 0$$
$$(\eta b_\eta)_\eta\delta\omega + \frac{1}{2}fb_\eta s_X\delta\omega - \frac{1}{2}f_\eta bs\delta\omega_X + \frac{1}{4}f_\eta bs\delta_X + \frac{1}{4\eta}fbs_X\delta\omega - \frac{1}{4\eta}b\delta\omega = 0$$
$$b^2\omega^2 = 2\eta a_\eta\kappa. \tag{41}$$

The boundary conditions have the form of conditions

(19). The system of equations (41) at $qPr = 2$ admits the solution

$$f = \frac{4\alpha\eta}{1+\alpha\eta}, \quad h = \frac{c_1}{(1+\alpha\eta)^{2Pr}}, \quad b = \lambda\frac{\sqrt{\alpha}\eta^{1/2}}{(1+\alpha\eta)^2}$$

$$a = -\frac{\lambda^2}{6}\frac{1}{(1+\alpha\eta)^3}, \quad s = X, \quad \sigma = X^{-1}$$

$$\delta = X^{-2}\left(1 \pm \frac{Gr}{Re^2}\frac{c_1^q}{2(3-q)\alpha^2}X^{3-q}\right)^{1/2}, \quad \omega = \delta^{1/2}/s$$

$$\kappa = \delta/s^2, \quad \alpha = \frac{3K_1}{16\pi}, \quad c_1 = \frac{(2Pr+1)Q_1}{8\pi},$$

$$\lambda = \frac{3\sqrt{\alpha}L_1}{8\pi}. \tag{42}$$

Then, the results of theoretical calculations of a swirled free-convection vertical jet can, eventually, be presented in the form

$$U = \frac{2\alpha}{(1+\alpha\eta)^2}\frac{1}{X}\left(1 \pm \frac{Gr}{Re^2}\frac{c_1^q}{2(3-q)\alpha^2}X^{3-q}\right)^{1/2}$$

$$\theta = \frac{c_1}{(1+\alpha\eta)^{2Pr}}\frac{1}{X},$$

$$W = \lambda\frac{\sqrt{\alpha}\eta^{1/2}}{(1+\alpha\eta)^2}\frac{1}{X^2}\left(1 \pm \frac{Gr}{Re^2}\frac{c_1^q}{2(3-q)\alpha^2}X^{3-q}\right)^{1/4}$$

$$P = -\frac{\lambda^2}{6}\frac{1}{(1+\alpha\eta)^3}\frac{1}{X^4}$$

$$\times\left(1 \pm \frac{Gr}{Re^2}\frac{c_1^2}{2(3-q)\alpha^2}X^{3-q}\right)^{1/2}. \tag{43}$$

These expressions are called the generalized self-similar solutions.

Now, different specific cases from formulae (43) derived will be considered. When $Gr/Re^2 \to 0$, expression (36) holds (the case of forced convection [1, 4]), and when $Gr/Re^2 \to \infty$, the self-similar solutions of free convection (20) are valid. It is also not difficult to show that approximate solution (39) is easily obtainable from exact solution (43). For this, it is necessary to write equation (43) in terms of variables (29) and expand it into a series in the parameter ε. Thus, series (39) converge into formula (43) at all

$$\left|\frac{Gr}{Re^2}\frac{c_1^q}{2(3-q)\alpha^2}X^{3-q}\right| \leqslant 1. \tag{44}$$

It follows from relations (43) that in the case of oppositely-directed free and forced convection flows there is a cross-section where the jet is smeared completely

$$X_T = \left(\frac{Gr}{Re^2}\frac{c_1^q}{2(3-q)\alpha^2}\right)^{1/q-3}. \tag{45}$$

In this case $U_T = 0$, $W_T = 0$, $P_T = 0$, and θ_T has the finite value. Thus the value of X_T depends not only on the Gr and Re numbers, but also on the initial momentum K_1, and excess enthalpy Q_1.

Some of the integral characteristics vary along the jet as:

$$K(X) = K_1\left(1 \pm \frac{Gr}{Re^2}\frac{c_1^q}{2(3-q)\alpha^2}X^{3-q}\right)^{1/2},$$

$$\dot{M}(X) = 8\pi X$$

$$E_1(X) = \frac{9}{80\pi^2}K_1^2X^{-1}\left(1 \pm \frac{Gr}{Re^2}\frac{c_1^q}{2(3-q)\alpha^2}X^{3-q}\right). \tag{46}$$

6. NON-SELF-SIMILAR SOLUTIONS

6.1. *An axisymmetric jet without allowance for the buoyancy forces*

It is assumed that a jet of viscous incompressible liquid ($W = 0$, $Gr/Re^2 = 0$) escapes from a finite-size nozzle. In this case, following ref. [4], the dimensionless stream function and temperature function are constructed in series form as

$$\psi(X,\eta) = \sum_{i=0}^{\infty} f_i(\eta)X^{1-i}, \quad \theta(X,\eta) = \sum_{i=0}^{\infty} h_i(\eta)X^{-1-i}. \tag{47}$$

Here

$$X = X, \quad \eta = Y^2/4X^2. \tag{48}$$

The substitution of relations (47) into equations (9)–(13), written in terms of new variables (48), and the equations of the terms at the equal powers of X will give an infinite system of differential equation to determine the unknown functions

$$(\eta f_0'')' + \frac{1}{2}(f_0f_0')' = 0, \quad \frac{1}{Pr}(\eta h_0')' + \frac{1}{2}(f_0h_0)' = 0$$

$$(\eta f_i'')' + \frac{1}{2}f_0f_i'' + \frac{1}{2}(i+2)f_0'f_i' - \frac{1}{2}(i-1)f_0''f_i = A_i$$

$$\frac{I}{Pr}(\eta h_i')' + \frac{1}{2}f_0h_i' + \frac{1}{2}(i+1)f_0'h_i = B_i$$

$$A_i = \sum_{j=1}^{i-1}\left\{\frac{1}{2}(j-1)f_jf_{i-j}'' - \frac{1}{2}[(i-j)+1]f_j'f_{i-j}'\right\},$$

$$B_i = -\frac{1}{2}f_i'h_0 + \frac{1}{2}(i-1)f_ih_0'$$

$$+\sum_{j=1}^{i-1}\left\{\frac{1}{2}(j-1)f_jh_{i-j}' - \frac{1}{2}[(i-j)+1]f_j'h_{i-j}\right\}. \tag{49}$$

A detailed discussion of the zero-order solution, f_0, h_0 [1, 17], and higher approximations, f_i, h_i ($i = 1, 2, 3$), can be found elsewhere [5] and therefore will not be considered here. Only the final results will be given.

These are

$$\psi = f_0' X - \gamma \sum_{i=0}^{2} \frac{1}{i+1}(2f_i'\eta + (i-1)f_i)X^{-i} + \cdots$$

$$U = \frac{1}{2}\left\{ f_0' X^{-1} - \gamma \sum_{i=0}^{2} \frac{1}{i+1}(2f_i''\eta + (i+1)f_i')X^{-2-i} + \cdots \right\}$$

$$\theta = h_0 X^{-1} - \gamma \sum_{i=0}^{2} \frac{1}{i+1}(2h_i'\eta + (i+1)h_i)X^{-2-i} + \cdots. \quad (50)$$

It is possible to approximate ψ, U, θ by higher-order terms ($i \geqslant 3$), but then new, undeterminable constants appear and the formulae become very cumbersome [5]. Here γ is the characteristic constant of the problem, which passes through the boundary and integral condition remaining undeterminable. Extending the analysis further, seek an exact non-self-similar solution. More general similarity transformations have the form

$$\psi(X,\xi) = f(\xi)s(X), \quad \theta(X,\xi) = h(\xi)\sigma(X)$$

$$X = X, \quad \xi = \frac{Y^2}{4}\delta(X). \quad (51)$$

Then, to determine the unknown functions, the following equations and boundary conditions are obtained

$$(\xi f_{\xi\xi})_\xi s\delta^2 + \frac{1}{2} ff_{\xi\xi} s\delta^2 s_X - \frac{1}{2} f_\xi^2 s\delta(s\delta)_X = 0$$

$$\frac{1}{Pr}(\xi h_\xi)_\xi \delta\sigma + \frac{1}{2} fh_\xi s_X \delta\sigma - \frac{1}{2} f_\xi hs\delta\sigma_X = 0$$

$$f(0) = 0, \quad \lim_{\xi\to 0}\sqrt{\xi} f_{\xi\xi} = 0, \quad f_\xi(\infty) = 0$$

$$\lim_{\xi\to 0}\sqrt{\xi} h_\xi = 0, \quad h(\infty) = 0. \quad (52)$$

System (52) admits the solution of the form

$$s(X) = X(1+\gamma/X), \quad \delta(X) = X^{-2}(1+\gamma/X)^{-2}$$

$$\sigma(X) = X^{-1}(1+\gamma/X)^{-1}. \quad (53)$$

In this case the form of the functions $f(\xi)$ and $h(\xi)$ coincides with the familiar solutions [1, 17]

$$f = \frac{4\alpha\xi}{1+\alpha\xi}, \quad h = \frac{c_1}{(1+\alpha\xi)^{2Pr}}, \quad \alpha = \frac{3K_1}{16\pi}$$

$$c_1 = \frac{(2Pr+1)Q_1}{8\pi}. \quad (54)$$

Relations (54) differ substantially from relations (36) by a more general expression

$$\xi = \frac{Y^2}{4X^2}\left(1+\frac{\gamma}{X}\right)^{-2}. \quad (55)$$

The final results are

$$U = \frac{1}{2} f_\xi(\xi)\frac{1}{X}\left(1+\frac{\gamma}{X}\right)^{-1}, \quad \theta = h(\xi)\frac{1}{X}\left(1+\frac{\gamma}{X}\right)^{-1}. \quad (56)$$

When $X \to \infty$, formulae (56) pass over into self-similar solutions [1, 17]. On having written relations (56) in terms of self-similar variables (48), approximate solution (50) can be obtained by the substitution of the McLauren series

$$U(X,\xi) = \frac{1}{2}\left\{ f_0' X^{-1} - \gamma \sum_{i=0}^{\infty} \frac{1}{i+1} \times (2f_i''\eta + (i+1)f_i')X^{-2-i} \right\}$$

$$\theta(X,\xi) = h_0 X^{-1} - \gamma \sum_{i=0}^{\infty} \frac{1}{i+1}(2h_i'\eta + (i+1)h_i)X^{-2-i}. \quad (57)$$

Here, the prime means differentiation with respect to η, equation (48); the quantities f_0, h_0 are defined by formulae (36). Consequently, relations (57) provide the assessment of the convergence region for the series obtained

$$|\gamma/X| < 1. \quad (58)$$

Moreover, the additional constants which appear in higher-order terms ($i \geqslant 3$) do not already have the sense of γ and are a general drawback of the asymptotic boundary layer scheme [1].

Note that solutions (57) satisfy the corresponding integral conditions, which in terms of variable (48) have the form

$$\int_0^\infty \left(2f_0' f_i' + \sum_{j=1}^{i-1} f_j' f_{i-j}' \right) d\eta = 0$$

$$\int_0^\infty \left(f_0' h_i + \sum_{j=1}^{i} f_j' h_{i-j} \right) d\eta = 0. \quad (59)$$

For the final solution of the problem, it now remains to determine the integration constant γ. It is characterized by the mass flow rate per second through the initial jet cross-section

$$\gamma = \dot{M}_0/8\pi, \quad \dot{M}_0 = \dot{M}(0), \quad \dot{M}(X) = 2\pi\int_0 UY\, dY. \quad (60)$$

Resuming the analysis, note that relations (56) and (60) can be considered also as a self-similar solution [1, 17] but which is shifted along the axis X. Physically, this operation means the 'recession' of the point of fictitious jet emergence, i.e. the source of the jet, into the nozzle to such an extent that the liquid flow rate at the nozzle tip, specified by the self-similar solution [17], would be the same as the actual flow rate.

The formal attempts to mathematically describe the flow in a laminar axisymmetric jet with the aid of a shifted self-similar solution [17] have long been known [18]. However, the question of the location of the fictitious source ('pole' of the jet) has not been resolved [19, 20].

6.2. *An axisymmetric jet with allowance for the buoyancy forces*

Again set the problem (9)–(15) ($W = 0$, $P = 0$). The solution is sought in the form of equation (51). For the function $f(\xi)$ and $h(\xi)$ there is the system of equations (52) (it is only necessary that the expression $\pm 2Gr/Re^2(h\sigma)^q$ be added to the first equation). Therefore, in order to obtain the non-self-similar solution, it is necessary that the finite nature of the source ($\dot{M}^0 \neq 0$) be taken into account:

$$U = \frac{2\alpha}{(1+\alpha\xi)^2}\frac{1}{X}\left(1+\frac{\gamma}{X}\right)^{-1} \times\left(1\pm\frac{Gr}{Re^2}\frac{c_1^q}{2(3-q)\alpha^2}(X+\gamma)^{3-q}\right)^{1/2}$$

$$\theta = \frac{c_1}{(1+\alpha\xi)^{2Pr}}\frac{1}{X}\left(1+\frac{\gamma}{X}\right)^{-1}, \quad \xi = \frac{Y^2}{4X^2}\left(1+\frac{\gamma}{X}\right)^{-2} \times\left(1\pm\frac{Gr}{Re^2}\frac{c_1^q}{2(3-q)\alpha^2}(X+\gamma)^{3-q}\right)^{1/2}$$

$$qPr = 2 \qquad (61)$$

The constants α, c_1, γ are defined by expressions (54) and (60). The merit of the formulae obtained lies in the possibility for the calculation of a vertical jet at the specified momentum K_1. Moreover, they allow the description of the damping characteristics of the jet issuing into a homogeneous quiescent medium as for the flow region downstream of the starting length, where the jet-like character of flow is most pronounced

$$U \propto \frac{1}{X}\left(1+\frac{\gamma}{X}\right)^{-1}, \quad \theta \propto \frac{1}{X}\left(1+\frac{\gamma}{X}\right)^{-1}$$

so at a great distance from the source where the jet behaves like a free-convective flow

$$U \propto X^{(1-q)/2}, \quad \theta \propto X^{-1}.$$

At $Gr/Re^2 = 0$, relations (61) pass over into expressions (56). It also follows from relations (61) that for each of the specified quantities Gr/Re^2, K_1, Q_1, $\dot{M}_0$ there is a point on the jet axis with the approach to which the jet thickness increases infinitely and the flow in the jet along the axis X comes to a halt:

$$X_T = \left(\frac{Gr}{Re^2}\frac{c_1^q}{2(3-q)\alpha^2}\right)^{1/q-3} - \frac{\dot{M}_0}{8\pi}. \qquad (62)$$

Note that the flow rate in the vertical jet ($qPr = 2$)

$$\dot{M}(X) = 8\pi(X+\gamma)$$

with a positive or negative buoyancy force turns to be equal to that in a non-convective jet.

6.3. *A swirled convective jet*

The results obtained in Sections 3–5 are valid for slightly swirled jets since in equation (9) the term $-\partial P/\partial X$ has been discarded, which, as is seen from solutions (16), (27), (28), (39), and (43), decreases with an increase of X much more rapidly than the remaining terms of this equation. Consequently, the formulae are valid only in the flow region where the assumption of the similarity between the profiles of the characteristic quantities of a swirled jet remains in force, i.e. these solutions have an asymptotic character and are valid only at a certain distance from the source. In this case it is natural that the higher the swirling, the greater is this distance, as compared to slightly swirled jets. But for 'moderate' swirling such a simplification is inadmissible, since, as the experiments show [2], the reverse streams originate in the axial region of the jet, with the axial velocity profiles on the jet axis being characterized by 'dips'. In this case the solution of the problem should be sought, as in ref. [4], in the form of asymptotic series in the negative powers of the distance from the source ($q = 1$)

$$\psi(X,\eta) = \sum_{i=0}^{\infty} f_i(\eta)X^{1-i}, \quad \theta(X,\eta) = \sum_{i=0}^{\infty} h_i(\eta)X^{-1-i}$$

$$W(X,\eta) = \sum_{i=0}^{\infty} b_i(\eta)X^{-3/2-i},$$

$$P(X,\eta) = \sum_{i+0}^{\infty} a_i(\eta)X^{-3-i}, \quad X = X, \quad \eta = Y^2/4X. \qquad (63)$$

The transformation of system (9)–(15) with the aid of relations (63) yields

$$(\eta f_0'')' + \frac{1}{2}f_0 f_0'' + 2\frac{Gr}{Re^2}h_0 = 0, \quad \frac{1}{Pr}(\eta h_0')' + \frac{1}{2}(f_0 h_0)' = 0, \quad (\eta b_0')' + \frac{1}{2}(f_0 b_0)' - \frac{1}{4\eta}b_0(1-f_0) = 0, \quad 2a_0'\eta = b_0^2$$

$$(\eta f_i'')' + \frac{1}{2}f_0 f_i'' + \frac{1}{2}if_0' f_i' - \frac{1}{2}(i-1)f_0'' f_i + 2\frac{Gr}{Re^2}h_i = A_i, \quad \frac{1}{Pr}(\eta h_i')' + \frac{1}{2}f_0 h_i' + \frac{1}{2}(i+1)f_0' h_i = B_i$$

$$(\eta b_i')' + \frac{1}{2}f_0 b_i' + \frac{1}{2}(i+1)f_0' b_i - \frac{1}{4\eta}b_i(1-f_0) = C_i, \quad 2\eta a_i' = D_i$$

$$A_i = \sum_{j=1}^{i-1}\left\{\frac{1}{2}(j-1)f_j f_{i-j}'' - \frac{1}{2}(i-j)f_j' f_{i-j}'\right\} - 2(a_{i-3}'\eta + ia_{i-3})$$

$$B_i = -\frac{1}{2}f_i' h_0 + \frac{1}{2}(i-1)f_i h_0' + \sum_{j=1}^{i-1}\left\{\frac{1}{2}(j-1)f_j h_{i-j}' - \frac{1}{2}[(i-j)+1]f_j' h_{i-j}\right\}$$

$$C_i = \sum_{j=1}^{i-1}\left\{\frac{1}{2}(j-1)f_jb'_{i-j} - \frac{1}{2}[(i-j)] + 1f'_jb_{i-j} + \frac{j-1}{4\eta}f_jb_{i-j}\right\} - \frac{1}{2}f'_ib_0 + \frac{1}{2}(i-1)f_ib'_0 + \frac{i-1}{4\eta}f_ib_0$$

$$D_i = 2b_0b_i + \sum_{j=1}^{i-1}b_jb_{i-j}, \quad \int_0^\infty\left(f'_0h_i + \sum_{j=1}^{i}f'_jh_{i-j}\right)d\eta = 0, \quad \int_0^\infty\left(f'_0b_i + \sum_{j=1}^{i}f'_jb_{i-j}\right)\eta^{1/2}\,d\eta = 0$$

$$(1-i)\int_0^\infty\left(2f'_0f'_i + \sum_{j=1}^{i-1}f'_jf'_{i-j} + 4a_{i-3}\right)d\eta = 4\frac{Gr}{Re^2}\int_0^\infty h_i\,d\eta. \tag{64}$$

Here, the prime means differentiation with respect to η. Note that the integration of equations (64) is performed by the method described in Section 4. Stop with the fourth term of the expansion of U, θ, W, P which will make it possible to assess the influence of the swirling degree on the velocity and temperature fields ($Pr = 2$)

$$U = \frac{1}{2}\left\{f'_0 - \gamma\sum_{i=0}^{2}\frac{1}{i+1}(f''_i\eta + if'_i)X^{-1-i} + \frac{\lambda^2}{8\alpha^2}(f''_0\eta + f'_0)X^{-3} + \cdots\right\}$$

$$\theta = h_0X^{-1} - \gamma\sum_{i=0}^{2}\frac{1}{i+1}(h'_i\eta + (i+1)h_i)X^{-2-i} + \frac{\lambda^2}{8\alpha^2}h'_0\eta X^{-4} + \cdots$$

$$W = b_0X^{-3/2} - \gamma\sum_{i=0}^{2}\frac{1}{i+1}\left[b'_i\eta + \left(i+\frac{3}{2}\right)b_i\right]X^{-5/2-i} + \frac{\lambda^2}{8\alpha^2}\left(b'_0\eta + \frac{1}{2}b_0\right)X^{-9/2} + \cdots$$

$$P = a_0X^{-3} - \gamma\sum_{i=0}^{2}\frac{1}{i+1}(a'_i\eta + (i+3)a_i)X^{-4-i} + \frac{\lambda^2}{8\alpha^2}(a'_0\eta + a_0)X^{-6} + \cdots. \tag{65}$$

Here, f_0, h_0, a_0, b_0 are defined by formulae (20) at $q = 1$ and $Pr = 2$. The approximation of U, θ, W, P by higher-order terms ($i > 4$) is also possible, but then expressions (65) acquire a cumbersome form and, which is more important, the functions f_i, h_i, a_i, b_i cannot be then expressed in terms of the self-similar solutions f_0, h_0, a_0, b_0, respectively. This means that it is impossible to obtain an exact non-self-similar solution for a swirled jet. The reason for this is that the assumption of the universality of velocity, pressure, etc. profiles in swirled jets is invalid near the source, while in the region of asymptotic trends, i.e. at some distance from the nozzle, W decreases more rapidly than U and V, and the flow tends to degeneration into a non-swirled one. However, expressions (65) allow a conclusion that the solution (63) has an intermediate asymptotic relation of the form

$$U \sim \frac{1}{2}f'_0\left[1 + \frac{3\lambda^2}{8\alpha^2}(X+\gamma)^{-3}\right]^{1/3}$$

$$\theta \sim h_0X^{-1}(1+\gamma/X)^{-1}$$

$$W \sim b_0X^{-3/2}(1+\gamma/X)^{-3/2}\left[1 + \frac{3\lambda^2}{8\alpha^2}(X+\gamma)^{-3}\right]^{1/6}$$

$$P \sim a_0X^{-3}(1+\gamma/X)^{-3}\left[1 + \frac{3\lambda^2}{8\alpha^2}(X+\gamma)^{-3}\right]^{1/3}$$

$$\eta \sim \frac{Y^2}{4X}(1+\gamma/X)^{-1}\left[1 + \frac{3\lambda^2}{8\alpha^2}(X+\gamma)^{-3}\right]^{1/3}.$$

which qualitatively correctly reflects the properties of the solution at large values of X. Of particular interest is the comparison of the results obtained, equations (65), with analogous formulae for a swirled jet ($Gr/Re^2 = 0$) [5]:

$$u = \frac{1}{2}\left\{f'_0x^{-1} - \gamma\sum_{i=0}^{1}\frac{1}{i+1}(2f''_i\eta + (i+1)f'_i)x^{-2-i} + \frac{\lambda^2}{48\alpha^2}7f''_0\eta + 5f'_0)x^{-3} + \cdots\right\}.$$

The effect of swirling on the flow (and heat transfer) is taken into account by the third and subsequent terms of asymptotic expansions for a jet without allowance for the buoyancy forces, while for a convective jet the effect of the jet swirling degree is allowed for starting from the fourth approximation (65). It indicates that the effect of swirling in vertical jets disappears much more rapidly than in jets without free convection, i.e. the circumferential velocity in jets with buoyancy forces decays more rapidly than the axial velocity. Therefore, at a certain distance along the flow, where the swirling scarcely exerts any effect at all, the swirled jets resemble the unswirled ones with buoyancy forces. Consequently, it can be expected that for free-convection swirled jets the following formulae ($Pr = 2$)

$$U = \frac{2\alpha}{(1+\alpha\xi)^2}\frac{1}{X}\left(1+\frac{\gamma}{X}\right)^{-1}\times\left(1 \pm \frac{Gr}{Re^2}\frac{c_1}{4\alpha^2}(X+\gamma)^2\right)^{1/2}$$

$$W = \frac{\sqrt{\alpha}\xi^{1/2}}{(1+\alpha\xi)^2}\frac{1}{X^2}\left(1+\frac{\gamma}{X}\right)^{-2}\times\left(1 \pm \frac{Gr}{Re^2}\frac{c_1}{4\alpha^2}(X+\gamma)^2\right)^{1/4}$$

$$\theta = \frac{c_1}{(1+\alpha\xi)^4}\frac{1}{X}\left(1+\frac{\gamma}{X}\right)^{-1}$$

$$P = -\frac{\lambda^2}{6}\frac{1}{(1+\alpha\xi)^3}\frac{1}{X^4}\left(1+\frac{\gamma}{X}\right)^{-4} \times\left(1\pm\frac{Gr}{Re^2}\frac{c_1}{4\alpha^2}(X+\gamma)^2\right)^{1/2}$$

$$\xi = \frac{Y^2}{4X^2}\left(1+\frac{\gamma}{X}\right)^{-2}\left(1\pm\frac{Gr}{Re^2}\frac{c_1}{4\alpha^2}(X+\gamma)^2\right)^{1/2} \qquad (66)$$

will give quite satisfactory results. Solution (66) is integration of the system of equations (41) with allowance for the finite nature of the source $\dot{M}_0 \neq 0$. Here, α, c_1, γ, λ are defined by formulae (42) and (60).

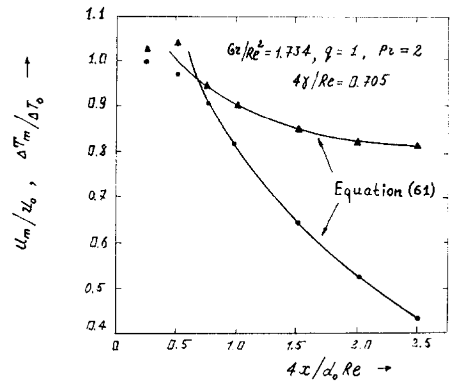

FIG. 3. Comparison with the results of work [21] (●, θ_m, ▲, U_m).

7. RESULTS AND DISCUSSION

Figure 2 illustrates the effect of buoyancy forces on the change of axial velocity U in the longitudinal direction at $\gamma = 10$ and at different values of the number $N = (Gr/Re^2)(c_1/4\alpha^2)$. As is seen, at small values of X the jet has the characteristics of flow without the buoyancy forces. With an increase in X, the characteristics of decay on the axis start to deflect from those corresponding to a jet with no buoyancy forces. With a further increase in X, the jet degenerates into an axisymmetric free-convective flow, with an increasing N bringing the region of this flow regime closer to the jet mouth. The curves of the maximum axial velocity and temperature are presented in Fig. 3. The points represent the values of U_m, θ_m, obtained by numerical calculation of equations (1)–(5) at $w = 0, \Delta p = 0, q = 1, Pr = 2$ [21]. It is interesting to note that, except for the data for the starting length, all of the results of numerical calculations [21] for the developed flow region are described by a single curve.

Now consider Fig. 4 with a family of curves representing different analytical solutions. As follows from this figure, the discrepancy between approximate (39) and generalized self-similar (43) solutions is observed at large values of X. Note that the convergence of series (39) slows down with an increase in the parameter

$$\frac{Gr}{Re^2}\frac{c_1}{4\alpha^2}X^2\left(\frac{Gr}{Re^2}\frac{c_1}{4\alpha^2}X^2 \to 1\right)$$

and, therefore, so that satisfactory results could be obtained, it is necessary to use a large number of terms of these series. This also indicates that the results of works studying the effect of free convection on the flow and heat transfer in jets by the perturbation method (2–3 terms of expansion) [22–24] should be used with caution. As follows from Fig. 4, the axial velocity at large values of X passes over into the relation $U \sim$ const., which corresponds to the self-similar solution for a laminar axisymmetric jet [8]. Therefore, the section X, where this law begins to hold, can be considered as the start of the region of free-convective flow in the jet. As to the effect of the buoyancy forces on the developing temperature profiles, the theoretical expressions derived do not depend on the number Gr/Re^2. This shows a weak dependence of temperature on the number Gr. In particular, the laws governing the decrease of θ_m in jets (with and without allowance for the buoyance forces) coincide ($\theta_m \propto X^{-1}$).

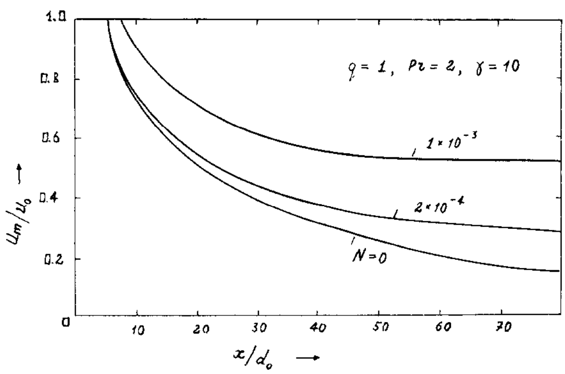

FIG. 2. The effect of buoyancy forces on the change in the maximum axial velocity.

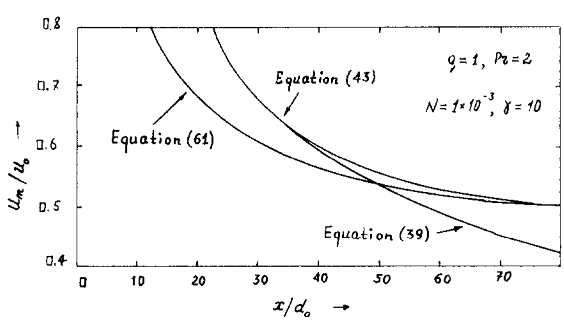

FIG. 4. Variation of the dimensionless axial velocity U depending on the dimensionless coordinate X.

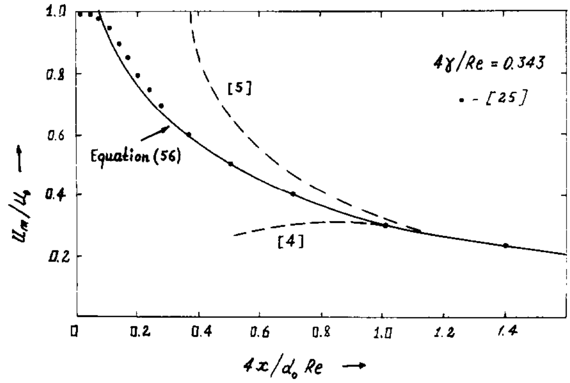

FIG. 5. Comparison of the present results with the results of other investigations [4, 5, 25].

Another important aspect of the problem of vertical jets is the determination of the jet deceleration cross-section X_T in the case when the buoyancy-induced momentum opposes the forced convection. The difference between expressions (45) and (62), used to determine X_T, lies in the fact that, when the finite nature of the source is allowed for, the value of X_T calculated by equation (62) turns to be smaller than that found by the self-similar approximation of problem (45).

Finally, in Fig. 5 a comparison is made between the axial velocities ($Gr/Re^2 = 0$, $W = 0$, $P = 0$) calculated by formula (56) of the present work and the numerical data presented in ref. [25] and analytical results of other authors [5].

8. CONCLUSION

Based on the laminar boundary-layer equations, with allowance for a new equation for density [12], the problem of vertical motion of a swirled jet under the action of initial momentum and buoyancy forces has been solved. The problem is shown to admit the self-similar solution valid for the free-convection flow zone. For the zone of mixed convection, an approximate self-similar solution is presented derived by the perturbation method. It is shown that these expressions constitute a particular case of the generalized self-similar solution obtained. Analogous relations have been earlier suggested in the MHD boundary-layer theory [26], but their derivation for the jet flows with buoyancy forces represents a new approach to their analysis. One of the basic results of the present analysis is the finding of the exact non-self-similar solution for both a vertical jet and jet without allowance for the buoyancy forces. A marked advantage of the formulae obtained is the possibility to calculate the vertical jets at the specified momentum in the initial cross-section and to allow for the finite nature of the source. For moderately swirled jets, based on the method of asymptotic expansions in negative powers of the distance from the jet source (up to the fourth-order terms), approximate non-self-similar solutions have been obtained which show that the swirling of jets exhibits, as compared to a non-convective jet, a comparatively weak effect on the flow and heat transfer in laminar buoyant jets. The formulae obtained are valid for two types of flows: when the direction of free convection-induced flow coincides with that of the vertical flow in a jet and when the directions are opposite. In the second case the results of calculation by the formulae exactly determine the location of the jet stagnation cross-section which is a function of the number Gr/Re^2, initial momentum, excess enthalpy and finite nature of the source.

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UN JET LAMINAIRE TOURNANT A CONVECTION NATURELLE

Résumé—On présente les résultats d'une analyse théorique de l'effet de la convection sur l'écoulement et le transfert thermique dans un panache vertical tourbillonnaire. On considère les cas de convection naturelle agissant favorablement ou défavorablement vis-à-vis de la convection forcée.

EIN LAMINARER VERWIRBELTER AUFTRIEBSSTRAHL

Zusammenfassung—Die Ergebnisse einer theoretischen Untersuchung werden dargelegt, die sich mit dem Einfluß der Konvektion auf die Strömung und den Wärmeübergang in einer senkrechten Wirbelfahne befaßt. Dazu werden die beiden Fälle betrachtet, daß die freie Konvektion gleich- oder gegensinnig zu der erzwungenen Strömung wirkt.

ЛАМИНАРНАЯ ЗАКРУЧЕННАЯ СТРУЯ СО СВОБОДНОЙ КОНВЕКЦИЕЙ

Аннотация—Представлены результаты теоретического анализа влияния свободной конвекции на течение и теплообмен в вертикальных закрученных струях. Рассмотрены случаи, когда свободная конвекция способствует и противодействует вынужденной конвекции.